Journal of Research in Mathematics Trends and Technology
Journal homepage: https://talenta.usu.ac.id/jormtt

Matrix Operators and *bv* Sequence Spaces

Khoirul Anwar[1], Supama[*2]
[1]Department of Mathematics, Universitas Gadjah Mada, Yogyakarta, 5522, Indonesia
[2]Department of Mathematics, Universitas Gadjah Mada, Yogyakarta, 5522, Indonesia
[*]Corresponding Author: supama@ugm.ac.id

ARTICLE INFO

Article history:
Received: 3 June 2024
Revised: 5 August 2024
Accepted: 7 September 2024
Available online: 30 September 2024

E-ISSN: 2656-1514
P-ISSN: -

How to cite:
Anwar, K., Supaman, "Matrix Operators and bv Sequence Spaces," Journal of Research in Mathematics Trends and, vol. V6, no. 2, Sep. 2024, doi: 10.32734/jormtt.v6i2.19935

ABSTRACT

In this paper, we study about the matrix domain of the bv sequence space by several triangle infinite matrices, namely the Cesaro mean of order one matrix, the *Generalized Weighted* matrix, and the Riesz matrix. Several things related to the matrix domain that will be observed include isomorphism of sequence spaces, BK-spaces, Schauder bases, α-duals, β-duals, γ-duals, and also the characterization of a mapping of a sequence space by some classes of matrix transformations. Furthermore, we also discuss inclusion relations of several matrix domains of the sequence space bv.

Keyword: Triangle Matrix, Matrix Domain, Schauder Basis, Dual Spaces.

1. INTRODUCTION

A theory of sequence spaces is a branch in mathematics that is growing quite rapidly and has applications in many fields, such as computation [1], crystallography [2], economics, and many others ([3], [4]). Therefore, the theory has received much attention from many researchers.

Research on sequence spaces can be carried out in several ways, such as by modifying their membership conditions or forming a new sequence space from a certain space by utilizing a certain function, matrix transformation, or any others. In this paper, we will discuss some sequence spaces that are formulated by utilizing an infinite transformation matrix. The such sequence spaces are called matrix domains [5]. In [5], Maddox discussed about the characterization of a mapping of a sequence space by some classes of the matrix transformations.

A matrix domain of sequence spaces is a very interesting topic in the theory of sequence spaces ([5], [6]). Apart from discussing the matrix domain, Malkowsky [6] also discussed a BK-space, that is a Banach space with continuous coordinate functionals. One knows that a BK-space is a key in matrix domain area. This is due to the fact that the BK-condition for any sequence space X gives characterization for the β-dual of the matrix domain of X [7].

Many researchers constructed a domain matrix of sequence spaces, such as bv and l_p sequences spaces, by using triangle infinite matrices. Some of the reasons are the existence of the inverse of triangle infinite matrix and the transformation of triangle matrix to any sequence. Some triangle infinite matrices that are often used include the Cesaro mean of order one, generalized weighted mean, and Riesz mean [8]. In [8], the characterization of the matrix transformation classes of the matrix domain of bv sequence spaces are discussed as well. In this paper, the terminology of the bv sequence space refers to the space of all sequence of bounded variation.

The bv sequence space is very interesting to discuss. In fact, the bv sequence space contains l_1, that is the space of all sequences of real numbers whose sums are absolutely convergent, and c, that is the space of all convergent sequences of real numbers. In addition, the bv sequence space is a matrix domain of l_1 sequence space which is obtained by using a certain triangle matrix.

Following the descriptions in the previous paragraphs, we discuss infinite matrix operators on the bv sequence spaces. In addition, we also examine the matrix domain of the bv sequence space and observe its properties.

As we know that any linear space has a corresponding space that is called a dual space. Dual vector spaces have important roles and applications in many branches of mathematics that use vector spaces. Particularly, when we apply to vector spaces of functions, dual spaces are used to describe measures, distributions, and Hilbert spaces. Consequently, the dual space is a very important concept in functional analysis. Let ω, cs, and bs be a set of all sequences of real numbers, a set of all convergent series, and a set of all bounded series, respectively. Following [8], the α-dual, the β-dual, and the γ-dual space of any sequence space X, each is denoted by $X^\alpha, X^\beta, X^\gamma$, are defined as follows:

$$X^\alpha = \{\bar{a} \in \omega : \ \bar{a}\bar{x} \in l_1, \qquad \text{for all } \bar{x} \in X\}$$

$$X^\beta = \{\bar{a} \in \omega : \ \bar{a}\bar{x} \in cs, \qquad \text{for all } \bar{x} \in X\} \text{ and}$$

$$X^\gamma = \{\bar{a} \in \omega : \ \bar{a}\bar{x} \in bs, \qquad \text{for all } \bar{x} \in X\}$$

Apart from being applicable to some fields mentioned above, of course dual spaces have many other uses. In this paper, we apply a dual space to characterize a mapping of a sequence space by some classes of matrix transformations.

2. METHOD

In this paper, the symbol ω is always meant as a collection of all sequences of real numbers. A norm space $X \subset \omega$ is called a BK-space if X is a Banach space and for every $n \in \mathbb{N}$, the canonical function $P_n : X \to \mathbb{R}$, where $P_n(\bar{x}) = x_n$, $\bar{x} = (x_n) \in X$ is continuous [6]. We also define the following sequence spaces:

$$bv = \left\{\bar{x} \in \omega : \ \sum_{k=1}^{\infty} |x_{k+1} - x_k| < \infty\right\} \text{ and } bv_0 = \left\{\bar{x} \in \omega : \ \lim_{k\to\infty} x_k = 0\right\}.$$

It is easy to check that both of them are Banach spaces with respect to the norm

$$\|\bar{x}\| = |x_1| + \sum_{k=1}^{\infty} |x_{k+1} - x_k|.$$

Study about sequence spaces is closely related to infinite matrices [9]. Let $A = (a_{nk})$ and $B = (b_{nk})$ be any infinite matrices. We define the multiplication AB as a matrix (c_{nk}), where

$$c_{nk} = \sum_{j=1}^{\infty} a_{nj}\, b_{jk}\ , \qquad (1)$$

whenever the summation in the right side of (1) exists for every $n \in \mathbb{N}$. An inverse of an infinite matrix $A = (a_{nk})$ is an infinite matrix $B = (b_{nk})$ such that $AB = I$, where

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

The matrix I is called an identity matrix. An infinite matrix $A = (a_{nk})$ is called a triangle matrix if for every $n \geq 1$, $a_{nn} \neq 0$ and $a_{nk} = 0$ for all $k > n$ [10]. Every triangle matrix has an inverse and its inverse is also a triangle matrix. Further, if A triangle and B is an inverse of A, then we can show that B is unique and $BA = AB = I$ [9].

Let A be any infinite matrix and $\bar{x} \in \omega$. We define $A(\bar{x}) = (A_n(\bar{x}))$, where $A_n(\bar{x}) = \sum_{k=1}^{\infty} a_{nk}\, x_k$ for every $n \geq 1$, provided $\sum_{k=1}^{\infty} a_{nk}\, x_k$ exists for all $n \geq 1$. In fact, for any triangle matrix A and any $\bar{x} \in \omega$, $A(\bar{x})$ exists [9].

Let X and Y be any sequence spaces, both are subsets of ω. An infinite matrix A defines a matrix transformation from X to Y, denoted by $A : X \to Y$, if for every $\bar{x} \in X$, $A(\bar{x})$ exists and $A(\bar{x}) \in Y$ [11]. It clear that any Matrix transformation is a linear operator [8]. A collection of all matrix transformation from X to Y denoted by $(X : Y)$.

The following lemmas and theorems can be found in [8].

Lemma 1 Let $X \subset \omega$ be a sequence space and A an infinite matrix. The set $X_A = \{\bar{x} \in \omega : A(\bar{x}) \in X\}$ is a normed space with respect to $\|\bar{x}\|_{X_A} = \|A(\bar{x})\|_X$.

Theorem 2.1 Let $X, Y \subset \omega$ be sequence spaces. If A is an infinite matrix and U is a triangle matrix such that UA exists, then $A \in (X : Y_U) \Leftrightarrow UA \in (X : Y)$.

Lemma 2 Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:

(i) $A \in (l_1 : l_\infty) \Leftrightarrow \sup_{k,n\geq 1} |a_{nk}| < \infty$.

(ii) $A \in (l_1 : c) \Leftrightarrow \sup_{k,n\geq 1} |a_{nk}| < \infty$ and for every $k \geq 1$ there are real number α_k such that $\lim_{n\to\infty} a_{nk} = \alpha_k$.

(iii) $A \in (l_1 : l_1) \Leftrightarrow \sup_{k\geq 1} \sum_{n\geq 1}^{\infty} |a_{nk}| < \infty$.

We also observe the following lemma.

Lemma 3 The space $(bv, \|\cdot\|_{bv})$ is isomorphic to the space $(l_1, \|\cdot\|_{l_1})$.

Proof. Notice that a matrix transformation $T : bv \to l_1$, where

$$T(\bar{x}) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}(\bar{x}), \qquad \bar{x} \in bv,$$

is an isomorphism.

3. RESULT AND DISCUSSION

In this section we discuss matrix domains of the sequence spaces by some matrix transformations. Let A be any infinite matrix. The matrix domain of bv by A, denoted by bv_A or $bv(A)$, is defined as

$$bv(A) = \{\bar{x} \in \omega : A(\bar{x}) \in X\}.$$

If A is a triangle matrix, then $bv(A)$ is a normed space with respect to $\|\bar{x}\|_{bv(A)} = \|A(\bar{x})\|_{bv}$.

For any triangle matrix A, the spaces $\left(bv(A),\ \|\cdot\|_{bv(A)}\right)$ and $(bv,\ \|\cdot\|_{bv})$ are isomorphic. It is proved in the following theorem.

Theorem 3.1 Let A be a triangle matrix. The space $\left(bv(A),\ \|\cdot\|_{bv(A)}\right)$ is isomorphic to $(bv,\ \|\cdot\|_{bv})$.

Proof. Since A a triangle matrix, then $A(\bar{x})$ exists. So, we can define a function $T : bv(A) \to bv$ by $T(\bar{x}) = A(\bar{x}),\ \bar{x} \in bv(A)$.
It is clear that T is a linear mapping. For any $\bar{x} \in bv(A)$, we have $\|T(\bar{x})\|_{bv} = \|A(\bar{x})\|_{bv} = \|\bar{x}\|_{bv(A)}$. It means that T is an isometri. Since A is a triangle matrix, then the inverse of A, i.e. A^{-1}, exists and it is also a triangle matrix [9]. For any $\bar{y} \in bv$, we can take $\bar{x} = A^{-1}(\bar{y})$ [9]. Since $A(\bar{x}) = \bar{y} \in bv$, then we obtain $\bar{x} \in bv(A)$ and $T(\bar{x}) = \bar{y}$. So, T is surjective. Thus, T is an isomorphism.

Following Lemma 3 and Theorem 3.1, we have the following corollary.

Corollary 3.1 For every triangle matrix A, $\left(bv(A),\ \|\cdot\|_{bv(A)}\right)$ is isomorphic to $\left(l_1,\ \|\cdot\|_{l_1}\right)$.

For any triangle matrix A, the transformation $A: bv(A) \to bv$ is continuous. It is shown in the following theorem.

Theorem 3.2 Let A be a triangle matrix. Then A is a continuous mapping from $\left(bv(A),\ \|\cdot\|_{bv(A)}\right)$ onto $(bv,\ \|\cdot\|_{bv})$.

Proof. Take any $\bar{x} \in bv(A)$ and any $\varepsilon > 0$. For any $\bar{y} \in bv(A)$,

$$\|A(\bar{x}) - A(\bar{y})\|_{bv} = \|A(\bar{x} - \bar{y})\|_{bv} = \|\bar{x} - \bar{y}\|_{bv(A)}\,.$$

By choosing $\delta = \varepsilon > 0$, we have $\|A(\bar{x}) - A(\bar{y})\|_{bv} < \varepsilon$, whenever $\|\bar{x} - \bar{y}\|_{bv(A)} < \delta$. These complete the proof.

Next, we prove that for any triangle matrix A, $bv(A)$ is a BK-space.

Theorem 3.3 If A is a triangle matrix, then $\left(bv(A),\ \|\cdot\|_{bv(A)}\right)$ is a BK-space.

Proof. By considering Theorem 3.1 and the completeness of bv, then $\left(bv(A),\ \|\cdot\|_{bv(A)}\right)$ is a Banach space. Therefore, the remain we have to show is the continuity of the canonical mapping $p_n: bv(A) \to \mathbb{R}$ for every $n \in \mathbb{N}$.

Take any $\bar{x}, \bar{y} \in bv(A)$, then $A(\bar{x} - \bar{y}) \in bv$. Since A is triangle matrix, then $A^{-1} = (b_{nk})$ exists, $A^{-1}(A(\bar{x} - \bar{y})) = \bar{x} - \bar{y}$, and $A(\bar{x} - \bar{y}) \in bv \subseteq l_\infty$. Further, for any integer $n \geq 1$,

$$|p_n(\bar{x}) - p_n(\bar{y})| = |x_n - y_n| = \left|\sum_{k=1}^{n} b_{nk}\, A_k(\bar{x} - \bar{y})\right| \leq \|A(\bar{x} - \bar{y})\|_\infty \sum_{k=1}^{n} |b_{nk}|$$

$$\leq \|\bar{x} - \bar{y}\|_{bv(A)} \sum_{k=1}^{n} |b_{nk}|\,.$$

Let $\varepsilon > 0$ be an arbitrary. By choosing $\delta = \frac{1}{\sum_{k=1}^{n}|b_{nk}|}\varepsilon > 0$, then we obtain $|p_n(\bar{x}) - p_n(\bar{y})| < \varepsilon$, whenever $\|\bar{x} - \bar{y}\|_{bv(A)} < \delta$. Thus, p_n is continuous for every natural number $n \geq 1$.

3.1 The matrix domain of bv induced by the Cesaro matrix

Let C be the Cesaro matrix of order 1, i.e. $C = (c_{nk})$ with $c_{nk} = \begin{cases} \frac{1}{n}, & 1 \leq k \leq n \\ 0, & k > n \end{cases}$. AS we mentioned before, the matrix domain of bv defined by using the triangle matrix C is $bv(C) = \{\bar{x} \in \omega : C(\bar{x}) \in X\}$. Note that if we take $\bar{x} = \left(\frac{(-1)^k}{k}\right)$ then $\bar{x} \notin bv$. However, since $C(\bar{x}) = \left(\frac{1}{n}\sum_{k=1}^{n} \frac{(-1)^k}{k}\right) \in bv$, then $\bar{x} \in bv(C)$. So, we have proved that $bv(C) \nsubseteq bv$.

Let C be a Cesaro matrix of order 1, i.e.

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & 0 & \cdots \\ 1/4 & 1/4 & 1/4 & 1/4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \text{ and } \Delta = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

If $\Phi = \Delta C$, then

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ -1+\frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ -\frac{1}{2}+\frac{1}{3} & -\frac{1}{2}+\frac{1}{3} & \frac{1}{3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Next, the inverse of the matrix C and Φ are C^{-1} and Φ^{-1} respectively, where

$$C^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 2 & 0 & 0 & \cdots \\ 0 & -2 & 3 & 0 & \cdots \\ 0 & 0 & -3 & 4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \text{ and } \Phi^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 0 & 0 & \cdots \\ 1 & 1 & 3 & 0 & \cdots \\ 1 & 1 & 1 & 4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

We recall what so called Schauder basis of any sequence space X (See for e.g. [8]). Let X be a sequence space. The sequence (b_k) in X is called a Schauder basis of X, if for each $\bar{x} \in X$ there exist a unique sequence (α_k) of scalars such that $\bar{x} = \sum_{k=1}^{\infty} \alpha_k \, b_k$.

Given a sequence $t^{(k)} = \left(t_n^{(k)}\right) \in bv(C)$ for all $k \geq 1$, where $t_n^{(k)} = 0 \ (1 \leq n < k)$, $t_n^{(k)} = (-1)^{n-k}\, k$ $(k \leq n \leq k+1)$, and $t_n^{(k)} = 0 \ (n > k+1)$. The following theorem shows that the sequence $(t^{(k)})$ is a Schauder basis of $bv(C)$.

Theorem 3.4 Given a sequence $t^{(k)} = \left(t_n^{(k)}\right) \in bv(C)$ for all $k \geq 1$, which is where $t_n^{(k)} = 0 \ (1 \leq n < k)$, $t_n^{(k)} = (-1)^{n-k}\, k \ (k \leq n \leq k+1)$, and $t_n^{(k)} = 0 \ (n > k+1)$. The sequence $(t^{(k)})$ is a Schauder basis of $bv(C)$.

Proof. Let $\bar{x} \in bv(C)$ be an arbitrary, then $C(\bar{x}) \in bv$. For each $k \geq 1$ defines $\alpha_k = C_k(\bar{x}) = \frac{1}{k}\sum_{j=1}^{k} x_j$. Note that for any fixed $k \geq 1$,

$$t^{(k)} = \left(t_1^{(k)}, t_2^{(k)}, t_3^{(k)}, \ldots\right) = \Big(\underbrace{0,0,0,\ldots,0}_{k-1}, k, -k, 0, 0, 0, \ldots\Big).$$

Then $C(t^{(k)}) = \Big(\underbrace{0,0,0,\ldots,0}_{k-1}, 1, 0, 0, 0, \ldots\Big) = e^{(k)}$. If for any natural number $m \geq 1$, we define $\bar{x}^{(m)} = \sum_{k=1}^{m} \alpha_k \, t^{(k)}$, then $\bar{x}^{(m)} \in bv(C)$. Further, $C\big(\bar{x} - \bar{x}^{(m)}\big) = \Big(\underbrace{0,0,0,\ldots,0}_{m}, \alpha_{m+1}, \alpha_{m+2}, \alpha_{m+3}, \ldots\Big)$, for every $m \geq 1$. Let $\varepsilon > 0$ be an arbitrary. Since $\sum_{k=1}^{n} |C_{k+1}(\bar{x}) - C_k(\bar{x})|$ convergent, then there exists a number $m_0 \in \mathbb{N}$ such that for each $m \in \mathbb{N}$ with $m \geq m_0$, we have $\sum_{k=m+1}^{\infty} |C_{k+1}(\bar{x}) - C_k(\bar{x})| < \varepsilon$. These imply

$$\left\|\bar{x} - \bar{x}^{(m)}\right\|_{bv(C)} = \left\|C(\bar{x} - \bar{x}^{(m)})\right\|_{bv} = \sum_{k=m+1}^{\infty} |\alpha_{k+1} - \alpha_k| = \sum_{k=m+1}^{\infty} |C_{k+1}(\bar{x}) - C_k(\bar{x})| < \varepsilon,$$

for every $m \geq m_0$. Thus, $\left\|\bar{x} - \bar{x}^{(m)}\right\|_{bv(C)} < \varepsilon$, for all $m \geq m_0$, then $\lim_{m \to \infty} \bar{x}^{(m)} = \bar{x}$. So,

$$\bar{x} = \lim_{m\to\infty} \bar{x}^{(m)} = \lim_{m\to\infty} \sum_{k=1}^{m} \alpha_k\, t^{(k)} = \sum_{k=1}^{\infty} \alpha_k\, t^{(k)}\,.$$

The uniqueness of the sequence (α_k) follows from the definition of $\left(t^{(k)}\right)$. Hence, the sequence $(t^{(k)})$ is a Schauder basis of $bv(C)$.

Note that for each $k \geq 1$, $C\left(t^{(k)}\right) = e^{(k)} \in c_0$. This means $t^{(k)} \in c_0(C)$. So, the sequence $(t^{(k)})$ is a Schauder basis of $bv_0(C) = bv(C) \cap c_0(C)$.

The Schauder basis of any sequence space is not unique. Another Schauder basis of $bv(C)$ is given in the following theorem.

Theorem 3.5 Given a sequence $t^{(k)} = \left(t_n^{(k)}\right) \in bv(C)$ for all $k \geq 1$, which is where $t_n^{(k)} = 0\ (1 \leq n < k)$, $t_n^{(k)} = k\ (n = k)$, and $t_n^{(k)} = 1\ (n > k)$. The sequence $(t^{(k)})$ is a Schauder basis of $bv(C)$.

Proof. Let $\bar{x} \in bv(C)$. By defining $\alpha_k = \Phi_k(\bar{x})$ for every $k \geq 1$, then the assertion follows.

Definition 1 For any $\bar{a} \in \omega$, we define the matrices $B = (b_{nk})$ and $D = (d_{nk})$ as the following:

(i) $b_{nk} = a_n$, if $1 \leq k < n$, $b_{nn} = n\, a_n$, and $b_{nk} = 0$, if $k > n$, and

(ii) $d_{nk} = k\, a_k + \sum_{j=k+1}^{n} a_j\ (1 \leq k < n)$, $d_{nk} = n\, a_n\ (k = n)$, and $d_{nk} = 0\ (k > n)$.

We also define

$$d_1 = \left\{\bar{a} \in \omega :\ \sup_{k\geq 1} \sum_{n=1}^{\infty} |b_{nk}| < \infty\right\}, \qquad d_2 = \left\{\bar{a} \in \omega :\ \lim_{n\to\infty} d_{nk} \text{ exist, for all } k \geq 1\right\}, \text{and}$$

$$d_3 = \left\{\bar{a} \in \omega :\ \sup_{k,n\geq 1} |d_{nk}| < \infty\right\}.$$

Further, we prove the following theorems.

Theorem 3.6 Let $B = (b_{nk})$ be a matrix as given in Definition 1. Then $[bv(C)]^{\alpha} = d_1$.

Proof. Let $\bar{x} \in bv(C)$, then $\bar{y} = \Phi(\bar{x}) \in l_1$. For each $n \geq 1$, we have

$$B_n(\bar{y}) = a_n \left(n\, y_n + \sum_{k=1}^{n-1} y_k\right) = a_n\, x_n\,.$$

So, $B(\bar{y}) = \bar{a}\bar{x}$. These imply, $\bar{a}\bar{x} \in l_1$ if and only if $B(\bar{y}) \in l_1$. As a consequence, we obtain that $\bar{a} \in [bv(C)]^{\alpha}$ if and only if $B \in (l_1 : l_1)$. Therefore, by Lemma 2, we get $\bar{x} \in [bv(C)]^{\alpha}$ if and only if $\sup_{k\geq 1} \sum_{n=1}^{\infty} |b_{nk}| < \infty$. This gives us a result that $[bv(C)]^{\alpha} = d_1$.

Theorem 3.7 Let $D = (d_{nk})$ be a matrix as given in Definition 1. The β-dual of the space $bv(C)$ is the set $d_2 \cap d_3$.

Proof. For any $\bar{x} \in bv(C)$, $\bar{y} = \Phi(\bar{x}) \in l_1$. It holds also that

$$D_n(\bar{y}) = \sum_{k=1}^{n} a_k \left(k\, y_k + \sum_{j=1}^{k-1} y_j\right) = \sum_{k=1}^{n} a_k\, x_k,$$

for every $n \geq 1$. So, $D(\bar{y}) = \left(\sum_{k=1}^{n} a_k\, x_k\right)$. Therefore, we get a statement that $\bar{a}\bar{x} \in cs$ if and only if $D(\bar{y}) \in c$, which implies $\bar{a} \in [bv(C)]^{\beta}$ if and only if $D \in (l_1 : c)$. Therefore, by Lemma 2. we find that $\bar{a} \in [bv(C)]^{\beta}$ if and only if $\lim_{n\to\infty} d_{nk}$ exist for all $k \geq 1$ and $\sup_{k,n\geq 1} |d_{nk}| < \infty$. This proves that $[bv(C)]^{\beta} = d_2 \cap d_3$.

Theorem 3.8 The γ-dual of $bv(C)$ is the set d_3.

Proof. Notice that for any $\bar{x} \in bv(C)$, $\bar{y} = \Phi(\bar{x}) \in l_1$. Since for each $n \geq 1$, $\sum_{k=1}^{n} a_k x_k = D_n(\bar{y})$, then $\bar{a}\bar{x} \in bs$ if and only if $D(\bar{y}) \in l_\infty$. This implies $\bar{a} \in [bv(C)]^\gamma$ if and only if $D \in (l_1 : l_\infty)$. Further, by following the Lemma 2. we find that $\bar{a} \in [bv(C)]^\gamma$ if and only if $\sup_{k,n\geq 1} |d_{nk}| < \infty$. Thus, $[bv(C)]^\gamma = d_3$.

The following theorem states inclusion relations in $bv_0(C)$.
Theorem 3.9 The following relations are true.

$$d_1 \subseteq [bv_0(C)]^\alpha, \qquad d_2 \cap d_3 \subseteq [bv_0(C)]^\beta, \qquad d_3 \subseteq [bv_0(C)]^\gamma.$$

Proof. Note that $bv_0(C) \subseteq bv(C)$. First, we will show that $[bv(C)]^\alpha \subseteq [bv_0(C)]^\alpha$. Let $\bar{a} \in [bv(C)]^\alpha$, then for each $\bar{x} \in bv(C)$ we have $\bar{a}\bar{x} \in l_1$. Especially, for each $\bar{x} \in bv_0(C)$ we have $\bar{a}\bar{x} \in l_1$, so $\bar{a} \in [bv_0(C)]^\alpha$. In the same way, it is obtained that $[bv(C)]^\beta \subseteq [bv_0(C)]^\beta$ and $[bv(C)]^\gamma \subseteq [bv_0(C)]^\gamma$. Since $[bv(C)]^\alpha = d_1$, $[bv(C)]^\beta = d_2 \cap d_3$ and $[bv(C)]^\gamma = d_3$, then the assertions follow.

Next, we discuss some characterizations of the matrix transformation classes of the space $bv(C)$, which are presented in the following theorems.
Theorem 3.10 Let Y be any sequence space and $A = (a_{nk})$ and $E = (e_{nk})$ infinite matrices such that $k, n \geq 1$ holds $e_{nk} = k\, a_{nk} + \sum_{j=k+1}^{\infty} a_{nj}$. The matrix transformation $A \in (bv(C) : Y)$ if and only if $(a_{nk})_k \in [bv(C)]^\beta$ for every $n \geq 1$ and $E \in (l_1 : Y)$.
Proof.
($\Longrightarrow$)Take any $\bar{x} \in bv(C)$. Since $A \in (bv(C) : Y)$, then $A(\bar{x})$ exists and $A(\bar{x}) \in Y$. Therefore, for each $n \geq 1$, $\sum_{k=1}^{\infty} a_{nk} x_k$ is convergent, so $(a_{nk})_k \in [bv(C)]^\beta$. Further, take any $\bar{y} \in l_1$, then there exists $\bar{x} \in bv(C)$, such that $\bar{y} = \Phi(\bar{x})$. For each $n \geq 1$, we have

$$A_n(\bar{x}) = \sum_{k=1}^{\infty} a_{nk} x_k = \sum_{k=1}^{\infty} a_{nk} \left(k\, y_k + \sum_{j=1}^{k-1} y_j \right) = \sum_{k=1}^{\infty} \left(k\, a_{nk} + \sum_{j=k+1}^{\infty} a_{nj} \right) y_k = \sum_{k=1}^{\infty} e_{nk}\, y_k$$
$$= E_n(\bar{y}).$$

Therefore, $A(\bar{x}) = E(\bar{y})$. This is followed by $E \in (l_1 : Y)$.
($\Longleftarrow$)Let $\bar{x} \in bv(C)$. Since for each $n \geq 1$, $(a_{nk})_k \in [bv(C)]^\beta$, then $\sum_{k=1}^{\infty} a_{nk} x_k$ is convergent, or in other words $A(\bar{x})$ exists. Since $A(\bar{x}) = E(\bar{y})$ for all $\bar{y} \in l_1$ and also $E \in (l_1 : Y)$, then we have $A(\bar{x}) \in Y$. So, we have proved $A \in (bv(C) : Y)$.
Theorem 3.11 Let $B = (b_{nk})$ and $F = (f_{nk})$ be infinite matrices such that $f_{1k} = b_{1k}$ and $f_{nk} = \frac{1}{n} b_{nk} - \frac{1}{n(n-1)} \sum_{j=1}^{n-1} b_{jk}$ for every $n \geq 2, k \geq 1$. Then for any sequence space Y, $B \in \big(Y : bv(C)\big) \Leftrightarrow F \in (Y : l_1)$.
Proof.
($\Longrightarrow$)Let $\bar{y} \in Y$. Since $B \in (Y : bv(C))$, then $B(\bar{y})$ exists and $B(\bar{y}) \in bv(C)$. By applying Φ, we get $\Phi\left(B(\bar{y})\right) \in l_1$. Note that $\Phi_1(B(\bar{y})) = \sum_{k=1}^{\infty} b_{1k}\, y_k = F_1(\bar{y})$ and

$$\Phi_n\left(B(\bar{y})\right) = \frac{1}{n} \sum_{k=1}^{\infty} b_{nk}\, y_k - \frac{1}{n(n-1)} \sum_{j=1}^{n-1} \sum_{k=1}^{\infty} b_{jk}\, y_k = F_n(\bar{y}),$$

for every $n \geq 2$. These imply $F(\bar{y}) = \Phi(B(\bar{y}))$, so $F(\bar{y} = \Phi\big(B(\bar{y})\big) \in l_1$. Thus, $F \in (Y : l_1)$.
($\Longleftarrow$)Let $\bar{y} \in Y$. Since $F \in (Y : l_1)$, then $F(\bar{y})$ exists and $F(\bar{y}) \in l_1$. Since $F(\bar{y}) = \Phi(B(\bar{y}))$ and $F \in (Y : l_1)$, then $\Phi\big(B(\bar{y})\big) \in l_1$. This implies $B(\bar{y}) \in bv(C)$.

3.2 The matrix domain of bv defined by the Riesz matrix

Let $q_k > 0$ for every $k \geq 1$. For any $k, n \geq 1$, we define $r_{nk} = \begin{cases} \frac{1}{Q_k} q_k, & 1 \leq k \leq n \\ 0, & k > n \end{cases}$, where $Q_k = \sum_{j=1}^{k} q_j$. The matrix $R = (r_{nk})$ is called a Riesz matrix or Riesz mean. If Ψ is a matrix defined by $\Psi = \Delta R$, then Ψ is a triangle matrix and

$$\Psi = \begin{pmatrix} \frac{1}{Q_1} q_1 & 0 & 0 & \cdots \\ \left(\frac{1}{Q_2} - \frac{1}{Q_1}\right) q_1 & \frac{1}{Q_2} q_2 & 0 & \cdots \\ \left(\frac{1}{Q_3} - \frac{1}{Q_2}\right) q_1 & \left(\frac{1}{Q_3} - \frac{1}{Q_2}\right) q_2 & \frac{1}{Q_3} q_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

We observe the following inclusion theorem.

Theorem 3.12 It is true that $bv(C) \subseteq bv(R)$ for some Riesz matrix R.

Proof. Let $\bar{x} \in bv(C)$ be an arbitrary, then $C(\bar{x}) = \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) \in bv$. For every $k \geq 1$, choose $q_k = 1$, then $Q_k = \sum_{j=1}^{k} 1 = k$. Furthermore, for the related Riesz matrix R, we find

$$R(\bar{x}) = \left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k\, x_k\right) = \left(\frac{1}{Q_n}\sum_{k=1}^{n} x_k\right) = C(\bar{x}) \in bv.$$

This implies $\bar{x} \in bv(R)$, hence $bv(C) \subseteq bv(R)$.

Generally, $bv(R) \not\subseteq bv$. Notice that $\bar{y} = \left(\frac{(-1)^k}{k}\right) \in bv(C)$. It can be shown that $\bar{y} \in bv(R)$, where R is the Riesz matrix as defined in the proof of Theorem 3.12. However, $\bar{y} \notin bv$.

Let $q_k > 0$ and $Q_k = \sum_{j=1}^{k} q_j$ for any integer $k \geq 1$. If for every integer $k \geq 1$, the sequence $t^{(k)} = \left(t_n^{(k)}\right)$ satisfies $t_n^{(k)} = 0\ (1 \leq n < k)$, $t_n^{(k)} = \frac{(-1)^{n-k}}{q_n} Q_k\ (k \leq n \leq k+1)$, and $t_n^{(k)} = 0\ (n > k+1)$, then $t^{(k)} \in bv(R)$ for every $k \geq 1$. It is easy to check that the sequence $\left(t^{(k)}\right)$ is a Schauder basis of $bv(R)$. We observe also that the sequence $\left(t^{(k)}\right)$, where $t_n^{(k)} = 0\ (1 \leq n < k)$, $t_n^{(k)} = \frac{1}{q_n} Q_n\ (n = k)$, and $t_n^{(k)} = \frac{1}{q_n}(Q_n - Q_{n-1})\ (n > k)$, is a Schauder basis of $bv(R)$.

Next, we will be formulated some characterizations of a matrix transformation classes of the space $bv(R)$.

Theorem 3.13 Let $q_k > 0$ and $Q_k = \sum_{j=1}^{k} q_j$ for all integer $k \geq 1$. If $A = (a_{nk})$ and $E = (e_{nk})$ are infinite matrices such that $e_{nk} = \frac{a_{nk}}{q_k} Q_k + \sum_{j=k+1}^{\infty} \frac{a_{nj}}{q_j}\left(Q_j - Q_{j-1}\right)$ for every $k, n \geq 1$, then for any sequence space Y, the following statements are equivalent.

(i) $A \in (bv(R) : Y)$.

(ii) $(a_{nk})_k \in [bv(R)]^{\beta}$ for every $n \geq 1$ and $E \in (l_1 : Y)$.

Proof.

$(i) \Longrightarrow (ii)$ Take any $\bar{x} \in bv(R)$. Since $A \in (bv(R) : Y)$, then $A(\bar{x})$ exist and $A(\bar{x}) \in Y$. Therefore, for every $n \geq 1$, $\sum_{k=1}^{\infty} a_{nk}\, x_k$ is convergent, so $(a_{nk})_k \in [bv(R)]^{\beta}$. Take any $\bar{y} \in l_1$, then there exists $\bar{x} \in bv(C)$ such that $\bar{y} = \Psi(\bar{x})$. Since for every $n \geq 1$,

$$A_n(\bar{x}) = \sum_{k=1}^{\infty} a_{nk}\, x_k = a_{n1}\frac{1}{q_1} Q_1 y_1 + \sum_{k=2}^{\infty} a_{nk}\left(\frac{1}{q_k}(Q_k - Q_{k-1})\sum_{j=1}^{k-1} y_j + \frac{1}{q_k} Q_k y_k\right) = \sum_{k=1}^{\infty} e_{nk}\, y_k = E_n(\bar{y}),$$

then $A(\bar{x}) = E(\bar{y})$. These imply $E(\bar{y}) \in Y$ for every $\bar{y} \in l_1$, i.e. $E \in (l_1 : Y)$.

$(ii) \Longrightarrow (i)$ Let $\bar{x} \in bv(R)$ be an arbitrary. Since for each $n \geq 1$ satisfy $(a_{nk})_k \in [bv(R)]^{\beta}$, then $\sum_{k=1}^{\infty} a_{nk}\, x_k$ is convergent, or in other words $A(\bar{x})$ exists. Since, $A(\bar{x}) = E(\bar{y})$ for all $\bar{y} \in l_1$ and $E \in (l_1 : Y)$, then $A(\bar{x}) \in Y$. So, $A \in (bv(R) : Y)$.

Theorem 3.14 Let $q_k > 0$ and $Q_k = \sum_{j=1}^{k} q_j$ for all integer $k \geq 1$. If $B = (b_{nk})$ and $F = (f_{nk})$ are infinite matrices such that $f_{1k} = \frac{1}{Q_1} q_1 b_{1k}$ and $f_{nk} = \left(\frac{1}{Q_n} - \frac{1}{Q_{n-1}}\right) \sum_{j=1}^{n-1} q_j\, b_{jk} + \frac{1}{Q_n} q_n\, b_{nk}$ for every $k, n \geq 2$, then for any sequence space Y,

$$B \in \big(Y : bv(R)\big) \Leftrightarrow F \in (Y : l_1).$$

Proof.
($\Longrightarrow$) Let $\bar{y} \in Y$. Since $B \in (Y : bv(R))$, then $B(\bar{y})$ exist and $B(\bar{y}) \in bv(R)$. Further, by applying Ψ, we find $\Psi\big(B(\bar{y})\big) \in l_1$. Notice that $\Psi_1\big(B(\bar{y})\big) = \frac{1}{Q_1} q_1 \sum_{k=1}^{\infty} b_{1k}\, y_k = F_1(\bar{y})$ and

$$\Psi_n\left(B(\bar{y})\right) = \left(\frac{1}{Q_n} - \frac{1}{Q_{n-1}}\right) \sum_{j=1}^{n-1} q_j \sum_{k=1}^{\infty} b_{jk}\, y_k + \frac{1}{Q_n} q_n \sum_{k=1}^{\infty} b_{nk}\, y_k = F_n(\bar{y}),$$

for every $n \geq 2$. So, $F(\bar{y}) = \Psi\big(B(\bar{y})\big) \in l_1$ for each $\bar{y} \in Y$. This implies $F \in (Y : l_1)$.
($\Longleftarrow$) Let $\bar{y} \in Y$. Since $F \in (Y : l_1)$, then $F(\bar{y})$ exist and $F(\bar{y}) \in l_1$. Since, $F(\bar{y}) = \Psi(B(\bar{y}))$ and $F \in (Y : l_1)$, then $\Psi\big(B(\bar{y})\big) \in l_1$. This implies $B(\bar{y}) \in bv(R)$. So, we obtain $B \in (Y : bv(R))$.

3.3. The matrix domain of bv with the Generalized Weighted matrix

Let u_k and v_k be non zero real numbers for all integer $k \geq 1$. The matrix $G = (g_{nk})$,

$$g_{nk} = \begin{cases} u_n v_k\,, & 1 \leq k \leq n \\ 0\,, & k > n \end{cases},$$

is called a generalized weighted matrix. If $\Gamma = \Delta G$, then

$$\Gamma = \begin{pmatrix} u_1 v_1 & 0 & 0 & \cdots \\ (u_2 - u_1) v_1 & u_2 v_2 & 0 & \cdots \\ (u_3 - u_2) v_1 & (u_3 - u_2) v_2 & u_3 v_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

By following the Theorem 3.1, we find the fact that $l_1 \cong bv \cong bv(C) \cong bv(R) \cong bv(G)$, i.e. the sequence spaces l_1, bv, $bv(C)$, $bv(R)$ and $bv(G)$ are isomorphic each other. In the next theorem, we prove the inclusion relationship between the sequence space bv, $bv(C)$, $bv(R)$ and $bv(G)$.
Theorem 3.15 For any Riesz matrix R, there exists a generalized weighted matrix G such that $bv(R) \subseteq bv(G)$.

Proof. Let q_k be any positive real number and $Q_k = \sum_{j=1}^{k} q_j$ for every $k \geq 1$. Suppose that R is a Riesz matrix related to (q_k) and (Q_k). Take any $\bar{x} \in bv(R)$, then $R(\bar{x}) = \frac{1}{Q_n} \sum_{k=1}^{n} q_k\, x_k \in bv$. By defining matrix $G = (g_{nk})$ by

$$g_{nk} = \begin{cases} \dfrac{q_k}{Q_k}\,, & 1 \leq k \leq n \\ 0\,, & k > n \end{cases},$$

then we obtain $G(\bar{x}) = (u_n \sum_{k=1}^{n} v_k\, x_k) = \left(\frac{1}{Q_n} \sum_{k=1}^{n} q_k\, x_k\right) = R(\bar{x})$. It means, $\bar{x} \in bv(G)$. So, the assertion that $bv(R) \subseteq bv(G)$ is proved.

Next, defines the space $bv_0(G) = \{\bar{x} \in \omega : G(\bar{x}) \in bv_0\}$ and $c_0(G) = \{\bar{x} \in \omega : G(\bar{x}) \in c_0\}$. Note that $bv_0(G) = bv(G) \cap c_0(G)$. Next, given definition of Schauder basis and will be shown two Schauder basis of $bv(G)$, which is presented in the following theorem.

Theorem 3.16 Suppose that for every $k \geq 1$, $u_k \neq 0$, $v_k \neq 0$, and $t^{(k)} = \left(t_n^{(k)}\right) \in bv(G)$ is a sequence defined by $t_n^{(k)} = 0 \ (1 \leq n < k)$, $t_n^{(k)} = \frac{(-1)^{n-k}}{u_k\ v_n} \ (k \leq n \leq k+1)$, and $t_n^{(k)} = 0 \ (n > k+1)$. Then the sequence $(t^{(k)})$ is a Schauder basis of $bv(G)$.

Proof. Take any $\bar{x} \in bv(G)$. For every $k \geq 1$, define $\lambda_k = G_k(\bar{x}) = u_k \sum_{j=1}^{k} v_j\, x_j$, then $\bar{x} = \sum_{k=1}^{\infty} \lambda_k\, t^{(k)}$. So, the sequence $(t^{(k)})$ is a Schauder basis of $bv(G)$.

Theorem 3.17 Suppose that for every $k \geq 1$, $u_k \neq 0$, $v_k \neq 0$, and $t^{(k)} = \left(t_n^{(k)}\right) \in bv(G)$ is a sequence defined by $t_n^{(k)} = 0 \ (1 \leq n < k)$, $t_n^{(k)} = \frac{1}{u_n\, v_n} \ (n = k)$, and $t_n^{(k)} = \frac{1}{v_n}\left(\frac{1}{u_n} - \frac{1}{u_{n-1}}\right) \ (n > k)$. Then the sequence $(t^{(k)})$ is a Schauder basis of $bv(G)$.

Proof. Let $\bar{x} \in bv(G)$. For every $k \geq 1$, choose $\alpha_k = \Gamma_k(\bar{x})$. By analog with Theorem 3.4, $\bar{x} = \sum_{k=1}^{\infty} \alpha_k\, t^{(k)}$. Hence, the sequence $(t^{(k)})$ is a Schauder basis of $bv(G)$.

In the next theorems, we present some characterizations of the matrix transformation classes of the space $bv(G)$.

Theorem 3.18 Let $u_k \neq 0$ and $v_k \neq 0$ be non zero real numbers for all $k \geq 1$, $A = (a_{nk})$ and $E = (e_{nk})$ infinite matrices such that $e_{nk} = \frac{a_{nk}}{u_k\, v_k} \sum_{j=k+1}^{\infty} \frac{a_{nj}}{v_j}\left(\frac{1}{u_j} - \frac{1}{u_{j-i}}\right)$ for every $k, n \geq 1$. For any sequence space Y,

$$A \in (bv(G): Y) \Leftrightarrow (a_{nk})_k \in [bv(G)]^{\beta} \text{ for every } n \geq 1 \text{ and } E \in (l_1 : Y).$$

Proof.

($\Rightarrow$) Let $\bar{x} \in bv(G)$, then $A(\bar{x})$ exist and $A(\bar{x}) \in Y$. Therefore, for every $n \geq 1$, it is true that $\sum_{k=1}^{\infty} a_{nk}\, x_k$ is convergent. So, we obtain that $(a_{nk})_k \in [bv(G)]^{\beta}$. Further, take any $\bar{y} \in l_1$. Then there exists $\bar{x} \in bv(G)$ such that $\bar{y} = \Gamma(\bar{x})$. Since,

$$A_n(\bar{x}) = \sum_{k=1}^{\infty} a_{nk}\, x_k = \frac{a_{n1}}{u_1 v_1} y_1 + \sum_{k=2}^{\infty} a_{nk} \left(\frac{1}{v_k}\left(\frac{1}{u_k} - \frac{1}{u_{k-1}}\right)\sum_{j=1}^{k-1} y_j + \frac{1}{u_k v_k} y_k\right) = \sum_{k=1}^{\infty} e_{nk}\, y_k = E_n(\bar{y})$$

for every $n \geq 1$, then $A(\bar{x}) = E(\bar{y})$. Since, $A(\bar{x}) \in Y$, then $E(\bar{y}) \in Y$. Hence, $E \in (l_1 : Y)$.

($\Leftarrow$) Let $\bar{x} \in bv(G)$. Since $(a_{nk})_k \in [bv(G)]^{\beta}$ for every $n \geq 1$, then $\sum_{k=1}^{\infty} a_{nk}\, x_k$ is convergent. Hence, $A(\bar{x})$ exists and $A(\bar{x}) = E(\bar{y}) \in l_1$. So, we have proved that $A \in (bv(G) : Y)$.

Theorem 3.19 Let $u_k \neq 0$ and $v_k \neq 0$ be non zero real numbers for all $k \geq 1$, $B = (b_{nk})$ and $F = (f_{nk})$ infinite matrices such that $f_{1k} = u_1\, v_1\, b_{1k}$ and $f_{nk} = (u_n - u_{n-1}) \sum_{j=1}^{n-1} v_j\, b_{jk} + u_n\, v_n\, b_{nk}$ for every $n \geq 2$. Then for any sequence space Y,

$$B \in \big(Y : bv(G)\big) \Leftrightarrow F \in (Y : l_1).$$

Proof.

($\Rightarrow$) Let $\bar{y} \in Y$. Since $B \in (Y : bv(G))$, then $B(\bar{y})$ exists and $B(\bar{y}) \in bv(G)$. This implies $\Gamma\left(B(\bar{y})\right) \in l_1$. Since, $\Gamma_1\left(B(\bar{y})\right) = u_1\, v_1 \sum_{k=1}^{\infty} b_{1k}\, y_k = F_1(\bar{y})$ and

$$\Gamma_n(B(\bar{y}) = (u_n - u_{n-1}) \sum_{j=1}^{n-1} v_j \sum_{k=1}^{\infty} b_{jk}\, y_k + u_n\, v_n \sum_{k=1}^{\infty} b_{nk}\, y_k = F_n(\bar{y})$$

for every $n \geq 2$, then $F(\bar{y}) = \Gamma\big(B(\bar{y})\big) \in l_1$. Hence, $F \in (Y : l_1)$.
($\Longleftarrow$) Let $\bar{y} \in Y$. Since $F \in (Y : l_1)$, then $F(\bar{y})$ exists and $F(\bar{y}) \in l_1$. Since $F(\bar{y}) = \Gamma(B(\bar{y}))$ and $F \in (Y : l_1)$, then we obtain $\Gamma\big(B(\bar{y})\big) \in l_1$. This implies $B(\bar{y}) \in bv(G)$. So, $B \in (Y : bv(G))$.

4. CONCLUSION

Some characterizations of the matrix domains of the bv sequence space that are defined by several triangle infinite matrices, namely the Cesaro mean of order one matrix, the *Generalized Weighted* matrix, and the Riesz matrix, have been formulated. Several things related to the matrix domain, such as isomorphism of sequence spaces, inclusion relations of several matrix domains of a sequence space, BK-spaces, Schauder bases, α-duals, β-duals, and γ-duals, are also deduced.

References

[1] M. S. Khan, B. A. Alamri, M. Mursaleen and Q. D. Lohani, "Sequence spaces m(0) and n(0) with application in clustering," *Journal of inequalities and applications,* vol. 1, 2017.

[2] E. Malkowsky, F. Ozger and V. Velickovic, "Some spaces related to cesaro sequence spaces and a application to crystallography," *MATCH Commun. Math. Comput. Chem.,* vol. 70, no. 3, pp. 867-884, 2013.

[3] O. Talo and F. Basar, "Certain spaces of sequences of fuzzy numbers defined by modulus function," *Demonstratio mathematica,* vol. 43, no. 1, pp. 139-150, 2010.

[4] M. S. Khan, M. Kaushal and Q. D. Lohani, "Cesaro paranormed sequence space based intuitionistic fuzzy distance measure," *Journal of Inequalities & Special Functions,* vol. 13, no. 1, 2022.

[5] I. J. Maddox, Elements of functional analysis, New York: Cambridge University Press, 1977.

[6] E. Malkowsky, "Recent results in theory of matrix transformations in sequence spaces," *MATEMATICKI VESNIK-BEOGRAD,* vol. 49, pp. 187-196, 1997.

[7] I. Djolovic and E. Malkowsky, "Applications of matrix domains of triangles in the characterization of summability factors," *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Seri A. Matematicas,* vol. 115, no. 4, p. 177, 2021.

[8] M. Kirisci, "The sequence space bv and some applications," *Matematica Aeterna*, vol. 4, no. 3, pp. 207-223, 2014.

[9] R. G. Cooke, Infinite matrices and sequence spaces, New York: Courier Corporation, 2014.

[10] A. Wilansky, Summability through functional analysis, New York: Elsevier, 2000.

[11] F. Basar, Summability theory and its applications, New York: Chapman and Hall/CRC, 2022.