

Vertex Exponent of Asymmetric Two-coloured Cycle

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Abstract. This paper is about an asymmetric two-coloured cycle. Let D be an asymmetric two-coloured cycle on n vertices, where n is odd and $n \geq 3$, we show that the exponent of the k th vertex of D is exactly $(n^2 - 1)/4 + \lfloor k/2 \rfloor$.

Keywords: Exponent Digraphs, Primitive, Two-coloured Digraphs, Vertex Exponent.

Abstrak. Paper ini membahas tentang cycle dwi-warna asimetrik. Misalkan D merupakan suatu cycle dwi-warna asimetrik dengan n vertex, dimana n ganjil dan $n \geq 3$, diperlihatkan bahwa eksponen vertex ke k dari D adalah tepat $(n^2 - 1)/4 + \lfloor k/2 \rfloor$.

Kata Kunci: Eksponen Digraph, Primitif, Digraph Dwi-warna, Eksponen Vertex.

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1 Digraphs and Two-coloured Digraphs

We discuss about vertex exponent of primitive asymmetric two-coloured digraphs with a special type. The notations and terminologies for digraphs and two-coloured digraphs in this paper is based on [1]. A *walk* of length m from vertex u to vertex v in a graph is a sequence of m of the form

$$(v_0, v_1), (v_1, v_2), \dots, (v_{m-1}, v_m)$$

where $v_0 = u$ and $v_m = v$. We state a walk w from u to v as a (u, v) -walk or w_{uv} and $\ell(w_{uv})$ is denoted by its length. A (u, v) -walk is *closed* provided $u = v$ and if not is *open*. A *path* from u to v is a walk with no repeated vertices except possibly $u = v$. A *cycle* is a closed path. A *loop* is a closed cycle of length 1.

A digraph D called *strongly connected* if for each pair of vertices u and v there is a (u, v) -walk and also a (v, u) -walk of length exactly k in D . The *exponent* of D , denoted by $\text{exp}(D)$, is the smallest of such positive integer k . A strongly connected digraph D is primitive if and only if the greatest common divisor of lengths of all cycle in D is 1 [1]. A *symmetric* digraph D is a digraph such that the arc (u, v) is in D whenever the arc (v, u) is in D . Since a symmetric digraph must have a cycle of length 2, a symmetric digraph is primitive if and only if it contains a cycle with odd length.

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A *two-coloured digraph* or a *2-digraph*, is a digraph in which each of its arcs is coloured by either red or blue (2 colours). In a two-coloured digraph we differentiate a walk by how many red and how many blue arcs it contains. We intend an $(h, k)^T$ -walk from u to v as a (u, v) -walk that consists of h red arcs and k blue arcs and the length is $h + k$. The vector $(r(w), b(w))^T$ is called the composition of the walk w . A two-coloured digraph D is *strongly connected* provided that its underlying digraph is strongly connected. Underlying digraph means the digraph obtained from D by ignoring its arc colour. An *asymmetric two-coloured digraph* is a symmetric two-coloured digraph for which an arc (u, v) is coloured by red whenever the arc (v, u) is coloured by blue and vice versa. A strongly connected two-coloured digraph is *primitive* provided there are nonnegative integers h and k such that for each pair of vertices u and v there is an $(h, k)^T$ -walk from u to v . The smallest nonnegative integer $h + k$ among all such nonnegative integers h and k is called the *2-exponent* of D , denoted by $\exp_2(D)$.

Let a two-coloured digraph D and $\{\delta = \delta_1, \delta_2, \dots, \delta_t\}$ as its set of all cycle. A *cycle matrix* M of D is a 2 by t matrix whose i th column is the composition of the cycle δ_i , $i = 1, 2, \dots, t$. That is

$$M = \begin{bmatrix} r(\delta_1) & r(\delta_2) & \dots & r(\delta_t) \\ b(\delta_1) & b(\delta_2) & \dots & b(\delta_t) \end{bmatrix}.$$

The content of M is defined to be 0 if the $\text{rank}(M) = 1$ and the greatest common divisor of the 2 by 2 minors of M , otherwise.

Shader and Suwilo [1] initiated the research on 2-exponents of two-coloured digraphs. They indicated that the largest 2-exponent of primitive two-coloured digraphs on n vertices lies on the interval $[(n^3 - 5n^2)/3, (3n^3 + 2n^2 - 2n)/2]$. Since that time, many papers have been published on the subject. Suwilo [2] has shown the exponent of an asymmetric primitive two-coloured (n, s) -lollipop. Let D be an asymmetric primitive two-coloured (n, s) -lollipop. Since D has a red path of length $(s + 1)/2 + (n - s)$, then $\exp_2(D) = (s^2 - 1)/2 + (s + 1)(n - s)$. Suwilo has also shown that if n is odd and $s = n$ or $s = n - 2$, then $\exp_2(D) = (n^2 - 1)/2$ and if n is even and $s = n - 1$, then $\exp_2(D) = n^2/2$. Gao and Shao [3] have shown the generalized exponent of primitive two-coloured Wielandt digraph $W_n^{(2)}$. They showed 3 types of Wielandt digraph and found the formula to find vertex exponent for each type. Fomichev and Avezova [4] found the exact formula for the exponents of the mixing digraphs of register transformations.

Suwilo [5] also found the vertex exponent of two-colored primitive extremal ministrong digraphs $D^{(2)}$ on n vertices. If $D^{(2)}$ has one blue arc, he showed that the exponents of vertices of $D^{(2)}$ lie on $[n^2 - 5n + 8, n^2 - 3n + 1]$. Since $D^{(2)}$ has two blue arcs, he also showed that the exponents of vertices in $D^{(2)}$ lie on $[n^2 - 4n + 4, n^2 - n]$. Vertex exponents of a class of two-colored of the Hamiltonian digraphs have been shown in [6]. They found that the vertex exponents of primitive two-colored digraph $L_n^{(2)}$ on $n \geq 5$ vertices whose underlying digraph is the Hamiltonian digraph consisting of the cycle $v_1 \rightarrow v_n \rightarrow v_{n-1} \rightarrow \dots \rightarrow v_2 \rightarrow v_1$ and the arc $v_1 \rightarrow v_{n-2}$ is known that $2n^2 - 6n + 2 \geq \exp(L_n^{(2)}) \geq (n^3 - 2n^2 + 1)/2$. If the $\exp(L_n^{(2)}) = (n^3 - 2n^2 + 1)/2$, then its vertex exponents lie on $[(n^3 - 2n^2 - 3n + 4)/4, (n^3 - 2n^2 + 3n + 6)/4]$ and if $\exp(L_n^{(2)}) = 2n^2 - 6n + 2$, then its vertex exponents lie on $[n^2 - 4n + 5, n^2 - 2n - 1]$.

The aim of this paper is to show the vertex exponent of an asymmetric primitive two-coloured cycle. We give the formula to find the vertex exponent of an asymmetric primitive two-coloured

cycle as the result.

2 Vertex Exponent of An Asymmetric Two-coloured Cycle

Let D be an asymmetric primitive two-coloured cycle of length n . If D is primitive, then n have to be odd and $n \geq 3$. Furthermore, if D is asymmetric, then D has directed cycles of length 2 with composition $(1, 1)^T$, and also has two directed cycles δ_1 and δ_2 of length n . The cycle are

$$(1, 2), (2, 4), \dots, (n - 1, n), (n, n - 2), \dots, (5, 3), (3, 1)$$

and

$$(1, 3), (3, 5), \dots, (n, n - 1), (n - 1, n - 3), \dots, (4, 2), (2, 1).$$

Hence, the compositions form of the directed cycles of D are $(1, 1)^T$, $(n - a, a)^T$, and $(a, n - a)^T$ for some nonnegative integer $0 \leq a \leq n$. This implies the cycle matrix of D is of the form

$$M = \begin{bmatrix} n - a & a & 1 & 1 & \dots & 1 \\ a & n - a & 1 & 1 & \dots & 1 \end{bmatrix}.$$

Since D is primitive, by [1] the content of M is 1. Therefore

$$1 = \gcd(n(n - 2a), n - 2a, 2a - n) = \pm(n - 2a).$$

This implies that $a = (n + 1)/2$ or $a = (n - 1)/2$, so without loss the generality we may assume that

$$M = \begin{bmatrix} (n + 1)/2 & (n - 1)/2 & 1 & 1 & \dots & 1 \\ (n - 1)/2 & (n + 1)/2 & 1 & 1 & \dots & 1 \end{bmatrix}.$$

We note that since D is asymmetric, every vertex of D lies on a $(1, 1)^T$ -walk. This means each $(r, b)^T$ -walk in D can be extended to a $(r + t, b + t)^T$ -walk for each positive integer $t \geq 1$.

Let D be an asymmetric primitive two-coloured cycle where the arcs colouring in succession are $(n + 1)/2$ red arcs and $(n - 1)/2$ blue arcs or vice versa. Since n is odd, there are two cases. For $n = 4m + 1$, colour γ_1 by giving red on (v_j, v_{j-2}) -arcs where $3 \leq j \leq (n + 1)/2$, (v_1, v_2) -arc, and (v_i, v_{i+2}) -arcs where $2 \leq i \leq (n - 1)/2$ and the others with blue. For $n = 4m + 3$, colour γ_1 by giving red on (v_j, v_{j-2}) -arcs where $3 \leq j \leq (n + 3)/2$, (v_1, v_2) -arc, and (v_i, v_{i+2}) -arcs where $2 \leq i \leq (n + 1)/2$ and the others with blue.

Lemma 2.1. *Suppose D be an asymmetric primitive two-coloured cycle of length n , then the exponent v_k of D $\exp_D(v_k) \geq \frac{1}{4}(n^2 - 1) + \lfloor \frac{k}{2} \rfloor$.*

Proof. Suppose D be an asymmetric primitive two-coloured cycle of length n . Since D is primitive, then n have to be odd and $n \geq 3$. Moreover, since D is asymmetric, for each v_k on D there is a $(1, 1)^T$ -closed walk from v_k to itself of length 2. Let $p_{k,x}$ be a path which connect v_k to v_x on D is a $(r_{p_{k,x}}, b_{p_{k,x}})^T$ -path consist $r_{p_{k,x}}$ red arcs and $b_{p_{k,x}}$ blue arcs. For each v_k on D , then $(r, b)^T$ -walk from v_k to v_x is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} r_{p_{k,x}} \\ b_{p_{k,x}} \end{bmatrix} + x_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + x_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers x_1, x_2 and x_3 .

Case 1. For $n = 4m + 1$, we show that $\exp_D(v_k) \geq \frac{1}{4}(n^2 - 1) + \lfloor \frac{k}{2} \rfloor$. If k is even, for each v_x on D , there is a $(r, b)^T$ -walk from v_k to v_x . We take a $(r, b)^T$ -walk from v_k to $v_{\frac{n-1}{2}}$ and to $v_{\frac{n+1}{2}}$, then we find $\exp_D(v_k) \geq \frac{1}{4}(n^2 - 1) + \frac{k}{2}$.

There are red path of length e and blue path of length f from v_k to $v_{\frac{n-1}{2}}$, then the $(r, b)^T$ -walk is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} + x_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + x_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers e, f, x_1, x_2 and x_3 .

There are also red path of length g and blue path of length h from v_k to $v_{\frac{n+1}{2}}$, then the $(r, b)^T$ -walk is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix} + y_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + y_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + y_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers g, h, y_1, y_2 and y_3 . We note that if $1 \leq k < \frac{n+1}{2}$, then $e = \frac{n-1}{4} - \frac{k}{2}$, $f = 0$, $g = 0$ and $h = \frac{n-1}{4} + \frac{k}{2}$. But if $\frac{n+1}{2} \leq k \leq n$, then $e = g = \frac{k}{2} - \frac{n-1}{4}$, $f = 0$, and $h = \frac{n-1}{2}$. By setting both of those equations we have

$$\begin{bmatrix} e - g \\ f - h \end{bmatrix} = (y_1 - x_1) \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + (y_2 - x_2) \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + (y_3 - x_3) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2.1}$$

Substraction the second by the first component of Equation 2.1, then we have $(y_1 - x_1) + (x_2 - y_2) = 2(\frac{n-1}{4})$. This implies $y_1 + x_2 \geq 2(\frac{n-1}{4})$ and hence $y_1 \geq \frac{n-1}{4}$ or $x_2 \geq \frac{n-1}{4}$. Hence we now have that $\exp_D(v_k) \geq \frac{1}{4}(n^2 - 1) + \frac{k}{2}$.

If k is odd, for each v_x on D , there is a $(r, b)^T$ -walk from v_k to v_x . We take a $(r, b)^T$ -walk from v_k to $v_{\frac{n-1}{2}}$ and to $v_{\frac{n+1}{2}}$, then we find $\exp_D(v_k) \geq \frac{1}{4}(n^2 - 1) + \frac{k-1}{2}$.

There are red path of length e and blue path of length f from v_k to $v_{\frac{n-1}{2}}$, then the $(r, b)^T$ -walk is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} + x_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + x_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers e, f, x_1, x_2 and x_3 .

There are also red path of length g and blue path of length h from v_k to $v_{\frac{n+1}{2}}$, then the $(r, b)^T$ -walk is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix} + y_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + y_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + y_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers g, h, y_1, y_2 and y_3 . We note that if $1 \leq k \leq \frac{n+1}{2}$, then $e = \frac{n-1}{4} + \frac{k-1}{2}$, $f = 0$, $g = 0$ and $h = \frac{n-1}{4} - \frac{k-1}{2}$. But if $\frac{n+1}{2} < k \leq n$, then $e = \frac{n-1}{2}$, $f = \frac{k-1}{2} - \frac{n-1}{4}$, $g = 0$, and $h = \frac{k-1}{2} - \frac{n-1}{4}$. By setting both of those equations we have

$$\begin{bmatrix} e - g \\ f - h \end{bmatrix} = (y_1 - x_1) \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + (y_2 - x_2) \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + (y_3 - x_3) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2.2}$$

Subtraction the second by the first component of Equation 2.2, then we have $(y_1 - x_1) + (x_2 - y_2) = 2(\frac{n-1}{4})$. This implies $y_1 + x_2 \geq 2(\frac{n-1}{4})$ and hence $y_1 \geq \frac{n-1}{4}$ or $x_2 \geq \frac{n-1}{4}$. Hence we now have that $\exp_D(v_k) \geq \frac{1}{4}(n^2 - 1) + \frac{k-1}{2}$. Then for each k where $1 \leq k \leq n$ we have $\exp_D(v_k) \geq \frac{1}{4}(n^2 - 1) + \lfloor \frac{k}{2} \rfloor$.

Case 2. For $n = 4m + 3$, we also show that $\exp_D(v_k) \geq \frac{1}{4}(n^2 - 1) + \lfloor \frac{k}{2} \rfloor$. If k is even, for each v_x on D , there is a $(r, b)^T$ -walk from v_k to v_x . We take a $(r, b)^T$ -walk from v_k to $v_{\frac{n+1}{2}}$ and to $v_{\frac{n+3}{2}}$, then we find $\exp_D(v_k) \geq \frac{1}{4}(n^2 - 1) + \frac{k}{2}$.

There are red path of length e and blue path of length f from v_k to $v_{\frac{n+1}{2}}$, then the $(r, b)^T$ -walk is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} + x_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + x_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers e, f, x_1, x_2 and x_3 .

There are also red path of length g and blue path of length h from v_k to $v_{\frac{n+3}{2}}$, then the $(r, b)^T$ -walk is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix} + y_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + y_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + y_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers g, h, y_1, y_2 and y_3 . We note that if $1 \leq k \leq \frac{n+1}{2}$, then $e = \frac{n-1}{2}$, $f = \frac{n+1}{4} + \frac{k}{2}$, $g = \frac{n+1}{4} - \frac{k}{2}$ and $h = \frac{n-1}{2}$. But if $\frac{n+1}{2} < k \leq n$, then $e = \frac{k}{2} - \frac{n-1}{4}$, $f = 0$, $g = 0$ and $h = \frac{3n-1}{4} - \frac{k}{2}$. By setting both of those equations we have

$$\begin{bmatrix} e - g \\ f - h \end{bmatrix} = (y_1 - x_1) \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + (y_2 - x_2) \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + (y_3 - x_3) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2.3}$$

Subtraction the second by the first component of Equation 2.3, then we have $(y_1 - x_1) + (x_2 - y_2) = 2(\frac{n-3}{4})$. This implies $y_1 + x_2 \geq 2(\frac{n-3}{4})$ and hence $y_1 \geq \frac{n-3}{4}$ or $x_2 \geq \frac{n-3}{4}$. Hence we now have that $\exp_D(v_k) \geq \frac{1}{4}(n^2 - 1) + \frac{k}{2}$.

If k is odd, for each v_x on D , there is a $(r, b)^T$ -walk from v_k to v_x . We take a $(r, b)^T$ -walk from v_k to $v_{\frac{n+1}{2}}$ and to $v_{\frac{n+3}{2}}$, then we find $\exp_D(v_k) \geq \frac{1}{4}(n^2 - 1) + \frac{k-1}{2}$.

There are red path of length e and blue path of length f from v_k to $v_{\frac{n+1}{2}}$, then the $(r, b)^T$ -walk is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} + x_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + x_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers e, f, x_1, x_2 and x_3 .

There are also red path of length g and blue path of length h from v_k to $v_{\frac{n+3}{2}}$, then the $(r, b)^T$ -walk is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix} + y_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + y_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + y_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers g, h, y_1, y_2 and y_3 . We note that if $1 \leq k \leq \frac{n+1}{2}$, then $e = \frac{3n-1}{4} + \frac{k-1}{2}$, $f = 0$, $g = 0$ and $h = \frac{k-1}{2} - \frac{n+1}{4}$. But if $\frac{n+1}{2} < k \leq n$, then $e = \frac{k-1}{2} - \frac{n+1}{4}$, $f = 0$, $g = 0$, and

$h = \frac{k-1}{2} - \frac{3n-1}{4}$. By setting both of those equations we have

$$\begin{bmatrix} e-g \\ f-h \end{bmatrix} = (y_1 - x_1) \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + (y_2 - x_2) \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + (y_3 - x_3) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2.4}$$

Subtraction the second by the first component of Equation 2.4, then if $1 \leq k \leq \frac{n+1}{2}$, we have $(y_1 - x_1) + (x_2 - y_2) = 2(\frac{n-3}{4})$. This implies $y_1 + x_2 \geq 2(\frac{n-3}{4})$ and hence $y_1 \geq \frac{n-3}{4}$ or $x_2 \geq \frac{n-3}{4}$. And if $\frac{n+1}{2} < k \leq n$, we have $(y_1 - x_1) + (x_2 - y_2) = 2(\frac{n-1}{4})$. This implies $y_1 + x_2 \geq 2(\frac{n-1}{4})$ and hence $y_1 \geq \frac{n-1}{4}$ or $x_2 \geq \frac{n-1}{4}$. Hence we now have that $\exp_D(v_k) \geq \frac{1}{4}(n^2 - 1) + \frac{k-1}{2}$. Then for each k where $1 \leq k \leq n$ we have $\exp_D(v_k) \geq \frac{1}{4}(n^2 - 1) + \lfloor \frac{k}{2} \rfloor$.

Lemma 2.2. Let D be an asymmetric primitive two-coloured cycle of length n , then the exponent v_k of D $\exp_D(v_k) \leq \frac{1}{4}(n^2 - 1) + \lfloor \frac{k}{2} \rfloor$.

Proof. Let D be an asymmetric primitive two-coloured cycle of length n . If D is primitive, then n have to be odd and $n \geq 3$. Moreover, if D is asymmetric, for each v_k on D there is a $(1, 1)^T$ -closed walk from v_k to itself of length 2. For $k = 1$, there is a $(r, b)^T$ -walk from v_1 to v_j with composition $(\frac{n^2-1}{8}, \frac{n^2-1}{8})^T$. We show that $\exp_D(v_1) = \frac{1}{4}(n^2 - 1)$. First, we show $\exp_D(v_1) \leq \frac{1}{4}(n^2 - 1)$. Let $P_{1,j}$ be a path which start from v_1 to v_j . There is a $(r, b)^T$ -walk from v_1 to v_j . In this case, the walk that starts at v_1 , moves to v_j along the path $P_{1,j}$, and moves $(r(p_{1,j}) - b(p_{1,j}))$ times around the cycle γ_1 is the shortest walk from v_1 to v_j . This composition of this walk is

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} r(p_{1,j}) \\ b(p_{1,j}) \end{bmatrix} + (r(p_{1,j}) - b(p_{1,j})) \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix}.$$

Notice that $P_{1,j}$ is a path from v_1 to v_j where $b(p_{1,j}) = 0$. Then the composition of its walk becomes

$$\begin{bmatrix} r \\ b \end{bmatrix} = (r(p_{1,j}) - b(p_{1,j})) \begin{bmatrix} \frac{n+1}{2} \\ \frac{n+1}{2} \end{bmatrix}.$$

We find that $P_{1,j} \leq \frac{n-1}{4}$. This implies $r(p_{1,j}) + b(p_{1,j}) \leq \frac{n-1}{4}$, hence $r(p_{1,j}) - b(p_{1,j}) \leq r(p_{1,j}) + b(p_{1,j}) \leq \frac{n-1}{4}$, then

$$\begin{bmatrix} r \\ b \end{bmatrix} \leq \begin{bmatrix} \frac{n^2-1}{8} \\ \frac{n^2-1}{8} \end{bmatrix}.$$

Now using $(1, 1)^T$ -walks, we can extend the walk into $(t, t)^T$ -walk with $t = \frac{n^2-1}{8}$ then

$$\exp_D(v_1) \leq \frac{1}{4}(n^2 - 1).$$

Next we show $\exp_D(v_1) \geq \frac{1}{4}(n^2 - 1)$. Let D be an asymmetric primitive two-coloured cycle of length n . Since D is primitive, then n must be odd and $n \geq 3$. Furthermore, since D is asymmetric, for v_1 on D there is a $(1, 1)^T$ -closed walk from v_1 to itself of length 2. Let $p_{1,j}$ be a path which connect v_1 to v_j on D is a $(r_{p_{1,j}}, b_{p_{1,j}})^T$ -path consist $r_{p_{1,j}}$ red arcs and $b_{p_{1,j}}$ blue arcs. For v_1 on D , then $(r, b)^T$ -walk from v_1 to v_j is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} r_{p_{1,j}} \\ b_{p_{1,j}} \end{bmatrix} + x_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + x_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers x_1, x_2 and x_3 .

For each v_j on D , there is a $(r, b)^T$ -walk from v_1 to v_j . If $n = 4m + 1$, we take a $(r, b)^T$ -walk from v_1 to $v_{\frac{n-1}{2}}$ and to $v_{\frac{n+1}{2}}$. And if $n = 4m + 3$, we take a $(r, b)^T$ -walk from v_1 to $v_{\frac{n+1}{2}}$ and to $v_{\frac{n+3}{2}}$. Then we find $\exp_D(v_k) \geq \frac{1}{4}(n^2 - 1)$.

There are red path of length e and blue path of length f from v_k to $v_{\frac{n-1}{2}}$ if $n = 4m + 1$ and to $v_{\frac{n+1}{2}}$ if $n = 4m + 3$, then the $(r, b)^T$ -walk is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} + x_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + x_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers e, f, x_1, x_2 and x_3 .

There are also red path of length g and blue path of length h from v_k to $v_{\frac{n+1}{2}}$ if $n = 4m + 1$ and to $v_{\frac{n+3}{2}}$ if $n = 4m + 3$, then the $(r, b)^T$ -walk is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix} + y_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + y_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + y_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers g, h, y_1, y_2 and y_3 . We note that if $n = 4m + 1$, then $e = \frac{n-1}{4}, f = 0, g = 0$ and $h = \frac{n-1}{4}$. But if $n = 4m + 3$, then $e = \frac{n-1}{2}, f = \frac{n+1}{4}, g = \frac{n+1}{4}$, and $h = \frac{n-1}{2}$. By setting both of those equations we have

$$\begin{bmatrix} e - g \\ f - h \end{bmatrix} = (y_1 - x_1) \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + (y_2 - x_2) \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + (y_3 - x_3) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2.5}$$

Substraction the second by the first component of Equation 2.5, then we have if $n = 4m + 1$, we have $(y_1 - x_1) + (x_2 - y_2) = 2(\frac{n-1}{4})$. This implies $y_1 + x_2 \geq 2(\frac{n-1}{4})$ and hence $y_1 \geq \frac{n-1}{4}$ or $x_2 \geq \frac{n-1}{4}$. And if $n = 4m + 3$, we have $(y_1 - x_1) + (x_2 - y_2) = 2(\frac{n-3}{4})$. This implies $y_1 + x_2 \geq 2(\frac{n-3}{4})$ and hence $y_1 \geq \frac{n-3}{4}$ or $x_2 \geq \frac{n-3}{4}$. Hence we now have that $\exp_D(v_k) \geq \frac{1}{4}(n^2 - 1)$. Since we have $\exp_D(v_k) \leq \frac{1}{4}(n^2 - 1)$ and also $\exp_D(v_k) \geq \frac{1}{4}(n^2 - 1)$ imply that $\exp_D(v_k) = \frac{1}{4}(n^2 - 1)$.

For $k \neq 1$, there is a $(r, b)^T$ -walk from v_k to v_j that starts from v_k to v_1 and then continue from v_1 to v_j . Let $p_{k,1}$ be a path from v_k to v_1 of length $\lfloor \frac{k}{2} \rfloor$. We have the composition of $(r, b)^T$ -walk from v_1 to v_j is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} \frac{n^2-1}{8} \\ \frac{n^2-1}{8} \end{bmatrix}.$$

Then we find the $(r, b)^T$ -walk from v_k to v_j is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} \leq \left\lfloor \frac{k}{2} \right\rfloor + \begin{bmatrix} \frac{n^2-1}{8} \\ \frac{n^2-1}{8} \end{bmatrix}.$$

Then we have $\exp_D(v_k) \leq \frac{1}{4}(n^2 - 1) + \lfloor \frac{k}{2} \rfloor$.

Theorem 2.3. Let D be an asymmetric primitive two-coloured cycle of length n , then the exponent v_k of D $\exp_D(v_k) = \frac{1}{4}(n^2 - 1) + \lfloor \frac{k}{2} \rfloor$.

Proof. Let D be an asymmetric primitive two-coloured cycle of length n . Since D is primitive, then n must be odd and $n \geq 3$. Lemma 2.1 and Lemma 2.2 imply that $\exp_D(v_k) = \frac{1}{4}(n^2 - 1) + \lfloor \frac{k}{2} \rfloor$.

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