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Some Vector *FK* Sequence Spaces Generated by Modulus Function

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Abstract. In this paper, some vector valued sequence spaces $\Gamma_f(X)$ and $\Lambda_f(X)$ using modulus function are presented. Furthermore, we examined some topological properties of these sequence spaces equipped with a paranorm.

Keywords: Modulus function, Paranorm, Vector valued sequence space.

Abstrak. *Pada paper ini, diperkenalkan beberapa ruang barisan bernilai vektor $\Gamma_f(X)$ dan $\Lambda_f(X)$ menggunakan fungsi modulus. Lebih lanjut, dipelajari beberapa sifat-sifat topologi dari ruang-ruang barisan ini dikenakan suatu paranorma tertentu.*

Kata Kunci: *Fungsi modulus, Paranorma, Ruang barisan bernilai vektor.*

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1 Introduction

Let X be a vector space and $\mathbb{R}$ be the set of real numbers. A function $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ is called modulus function if following condition of f satisfying:

1. f is vanishing at zero
2. f satisfies triangle inequality
3. f is an increasing function i.e. $f(\cdot) \uparrow$
4. f is a continuous function from the right at 0 [1]

The function f must be continuous for every element x in $(0,\infty)$. The space of all real number sequences (x_n) such that the infinite series of absolute modulus function is finite denoted by $\ell(f)$ [2]

$$\sum_{n=1}^{\infty} f(|x_n|) < \infty.$$

The space $\ell(f)$ becomes a FK-space under the F-norm

$$p(x) = \sum_{n=1}^{\infty} f(|x_n|) < \infty.$$

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Adnan [3] examined the FK-space properties of an analytic and entire real sequence space using modulus function. He showed the characterization to matrix transformation of Ruckle's space $\ell(f)$ into analytic FK-space. For the theory of FK-space we refer to Banas and Mursaleen [4].

Through the article $\Omega(X)$, $\Gamma_f(X)$, $\Lambda_f(X)$ denoted by the space of vector value sequences, entire vector value sequence space and analytic vector value sequence space. The vector value sequence space studied by some authors [5, 6, 7, 8, 9, 10, 11, 12, 13]. Further, the concept of sequence space using modulus function was investigated by [14, 15, 16, 17, 18].

Recently, Herawati [5] studied the geometric of the vector value sequence spaces defined by order-φ function under Lattice norm. Further, Gultom [6] studied some topologies properties of a finite arithmatic mean vector value sequence space denoted by $W_f(X)$ for X is a linear space and f is a φ-function.

A functional is called paranormed if satisfies the properties $p : X \to \mathbb{R}$ that satisfies the properties $p(\theta) = 0$, with θ is the zero vector in X, non-negative, p satisfies triangle inequalities, even and every real sequence (λ_n) with $|\lambda_n - \lambda| \to 0$. The space X with paranorm p is called *paranormed space*, written as $X = (X, p)$ [1, 19].

In this work, we define the space of vector value sequences $\Gamma_f(X)$ and $\Lambda_f(X)$ called entire and analytic vector valued sequence spaces generated by modulus function and study the topological properties of the sets equipped with paranorm.

2 Main Results

In this main result section, firstly, we introduce paranorm on this space and examine some topological properties such as complete properties. Let X be a Banach space and f be a modulus function. Let $y(n) = f(\|x(n)\|_X) \in \mathbb{R}$ for all natural numbers n, then we get a sequence $y = (y(n))$. We define the sets

$$\Gamma_f(X) = \left\{ x = (x(n))_{n\in\mathbb{N}} : x(n) \in X \text{ and } (y(n))^{\frac{1}{n}} \to 0, n \to \infty \right\}$$

$$\Lambda_f(X) = \left\{ x = (x(n))n \in \mathbb{N} : x(n) \in X \text{ and } \sup_{n\in\mathbb{N}} \{(y(n))^{\frac{1}{n}}\} < \infty \right\}$$

Theorem 1.
The sets $\Gamma_f(X)$ and $\Lambda_f(X)$ are vector spaces.
Proof.
Let x, z be any elements in $\Gamma_f(X)$, then

$$\lim_{n\to\infty} (y(n))^{\frac{1}{n}} = 0 \text{ and } \lim_{n\to\infty} (w(n))^{\frac{1}{n}} = 0$$

for $n \to \infty$, with $y(n) = f(x(n)$ and $w(n) = f(z(n))$ for each natural number n. We will apply the following inequality : if $a_n, b_n \in \mathbb{R}$ and $0 \leq q_n \leq \sup q_n = H$ for each natural number n, then

$$|a_n + b_n|^{q_n} \leq M(|a_n|^{q_n}) + |b_n|^{q_n}$$

where $M = max\{1, 2^{H-1}\}$. Therefore,

$$(y(n)+w(n))^{\frac{1}{n}} \leq (y(n))^{\frac{1}{n}} + (w(n))^{\frac{1}{n}}$$

Since $(q_n) = (\frac{1}{n})$, then $H = \sup \frac{1}{n} = 1$. Thus

$$(y(n)+w(n))^{\frac{1}{n}} \leq (y(n))^{\frac{1}{n}} + (w(n))^{\frac{1}{n}}$$

Since $(y(n))^{\frac{1}{n}} \to 0$ and $(w(n))^{\frac{1}{n}}$ for $n \to \infty$, then $(y(n)+w(n))^{\frac{1}{n}} \to 0$ for $n \to \infty$. Therefore, we obtain $x+y \in \Gamma_f(X)$. Further, for element $x \in \Gamma_f(X)$ and $\alpha \in \mathbb{R}$, then

$$(y(n))^{\frac{1}{n}} \to 0, n \to \infty$$

Because of an increasing function f and the positivity of $|\alpha|$, then from the Archimedean properties, there exists natural number n_0 with

$$f(|\alpha|\|x(n)\|) \leq f(2^{n_0}\|x(n)\|)$$

Since f satisfies $\triangle_2$-condition, we get

$$(f(2^{n_0}\|x(n)))^{\frac{1}{n}} = K^{\frac{n_0}{n}}(f(\|x(n)\|))^{\frac{1}{n}} \to 0$$

for each natural number n. It shows that $\alpha x \in \Gamma_f(X)$. Because $x+z \in \Gamma_f(X)$ and $\alpha x \in \Gamma_f(X)$ for each $x, y \in \Gamma_f(X)$ and each $\alpha \in \mathbb{R}$, we get $\Gamma_f(X)$ is a vector or linear space and the proof of the theorem is finished. In the same way, it can be shown that $\Lambda_f(X)$ is a vector space. ∎

Theorem 2.
A functional $p : \Gamma_f(X) \to \mathbb{R}$ defined by

$$p(x) = \sup_{n\geq 1}\left\{(y(n))^{\frac{1}{n}}\right\}$$

is a paranorm.

Proof.
Let x be an element in $\Gamma_f(X)$. It is clear that the functional p is non-negative, $p(\theta) = 0$, with θ is the zero vector in X and even, for each $x \in \Gamma_f(X)$. Now, we will show that p satisfies the triangle inequality. To do that, take any $x, z \in \Gamma_f(X)$, then

$$\lim_{n\to\infty}(y(n))^{\frac{1}{n}} = 0 \text{ and } \lim_{n\to\infty}(w(n))^{\frac{1}{n}} = 0$$

for $n \to \infty$, with $y(n) = f(x(n))$ and $w(n) = f(z(n))$ for each $n \in \mathbb{N}$. we obtain

$$\sup\left\{(y(n)+w(n))^{\frac{1}{n}}\right\} \leq \sup\left\{(y(n))^{\frac{1}{n}} + (w(n))^{\frac{1}{n}}\right\}$$

Therefore, there's vector sequences of $x, y \in \Gamma_f(X)$, we get p satisfies the triangle inequality. Next, we will show that p satisfies the continuity of scalar multiplication. To do that, take any real sequence (λ_n) and $(x(n)) \in \Gamma_f(X)$ with $|\lambda_n - \lambda| \to 0$ for $n \in \infty$. We have

$$\begin{aligned}(f(\|x(n))\|_X))^{\frac{1}{n}} &= (f(\|\lambda_k x(n) - \lambda x(n)\|))^{\frac{1}{n}} \\ &= (f(\|(\lambda_n - \lambda)x(n)\|) + \|\lambda(x(n) - x)\|))^{\frac{1}{n}} \\ &\leq ((f|\lambda_n - \lambda|\|x(n)\| + |\lambda|\|(x(n) - x)\|)^{\frac{1}{n}}\end{aligned}$$

and

$$\begin{aligned}p(\lambda_n x(n) - \lambda x(n)) &= \sup\{(f(\|\lambda_n x(n) - \lambda x(n)\|))^{\frac{1}{n}}\} \\ &\leq |\lambda_n - \lambda| p(x(n) + |\lambda| p(x(n) - x) \to 0\end{aligned}$$

Hence, $p(\lambda_n x(n) - \lambda x(n)) \to 0$. The proof of the theorem is finished. ■

Theorem 3.
The vector spaces of $\Gamma_f(X)$ and $\Lambda_f(X)$ are complete paranormed sequence space under the paranorm defined in Theorem 2.
Proof.
Take any Cauchy sequence (x^i) in $\Gamma_f(X)$ with $x^i = (x^i(n)) = (x^i(1), x^i(2), ...,)$. Therefore, for any positive real number ε, there exists $i_0 \in \mathbb{N}$, for all $j \geq i \geq i_0$, we get

$$p(x^j - x^i) = \sup\left\{(f(\|x^j(n) - x^i(n)\|))^{\frac{1}{n}}\right\} < \varepsilon$$

Since $\sup\,(f\|x^j(n) - x^i(n)\|))^{\frac{1}{n}} < \varepsilon$, we have $(f(\|x^j(n) - x^i(n)\|))^{\frac{1}{n}} < \varepsilon$ for $\varepsilon > 0$. Since f is a modulus function, then $\|x^j(n) - x^i(n)\| = 0$ for each natural number n. In other words, $\|x^j(n) - x^i(n)\| < \varepsilon$. It shows that for each natural number n of the sequence $(x^j(n))$ is a Cauchy. Since X is a complete normed space, then $(x^j(n))$ converges to $x(n) \in X$. Hence, $\lim_{j\to\infty} x^j(n) = x(n)$ for all n. Therefore, there's sequence $x = (x(n)) = (x(1), x(2), ...,)$ such that

$$\begin{aligned}\sup\left\{(f(\|x - x^i\|))^{\frac{1}{n}}\right\} &= \sup\left\{(f(\|\lim_{i\to\infty} x - x^i\|))^{\frac{1}{n}}\right\} \\ &= \sup\left\{\lim_{i\to\infty}(f(\|x - x^i\|))^{\frac{1}{n}}\right\} \\ &= \lim_{i\to\infty}\sup\left\{(f(\|x - x^i\|))^{\frac{1}{n}}\right\}\end{aligned}$$

for every $i \geq i_0$. By using the definition of paranorm, we get

$$p(x - x^i) = \sup\left\{(f(\|x - x^i\|))^{\frac{1}{n}}\right\} < \varepsilon$$

It shows that $x^i \to x$ for $i \to \infty$. Then it will be shown that $x \in \Gamma_f(X)$. Using the continuous property of f, we get

$$\begin{aligned}(f(\|x\|))^{\frac{1}{n}} &= (f(\|\lim_{i\to\infty} x^i\|))^{\frac{1}{n}} \\ &= \lim_{i\to\infty}(f(\|x^i\|))^{\frac{1}{n}} \to 0\end{aligned}$$

for $i \to \infty$. Hence, $x \in \Gamma_f(X)$. The proof of this theorem is finished. ■

3 Conclusions

According to the main results, it can be concluded $\Gamma_f(X)$ and $\Lambda_f(X)$ are complete paranormed sequence space under the paranorm.

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